# Non-abelian statistics and the $S$ matrix

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Non-abelian statistics are just plain interesting.

They probably occur in the $\nu = 5/2$ FQHE, and people are constructing time-reversal-invariant models which realize them.

One conceivable application is in quantum computing.

In 2+1 dimensions, the statistics of particles follows from the properties of the wavefunction under braiding.

With anyons, the wavefunction can change by a phase.

With non-abelian statistics, how the wavefunction changes depends on the order in which the particles are braided.

Outline:

1. projecting onto the plane

2. what this has to do with the $S$ matrix

3. finding field theories

4. finding lattice models

work with E. Fradkin

related work with E. Ardonne and E. Fradkin

A convenient way of describing braiding is to project the world line of the particles onto the plane. Then the braids become overcrossings

$$B = \Large{\times}$$

and undercrossings

$$B^{-1} = \Large{\times}$$

The generators of the braid group must satisfy

$$B_i B_{i+1} B_i = B_{i+1} B_i B_{i+1}$$

time

For non-abelian statistics, we need the $B_i$ to be matrices so, e.g., $B_i B_{i+1} \neq B_{i+1} B_i$.

One simple example is well-known from knot theory. Let $e$ be a monoid:

$$I \qquad\qquad e$$
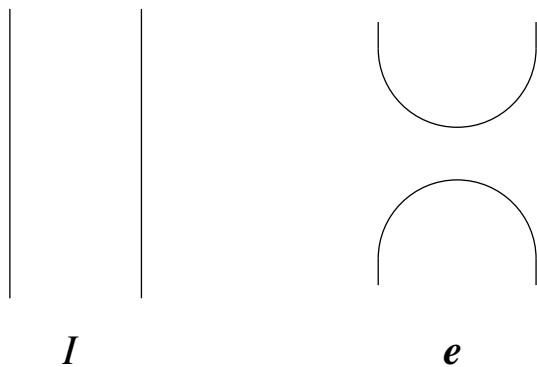
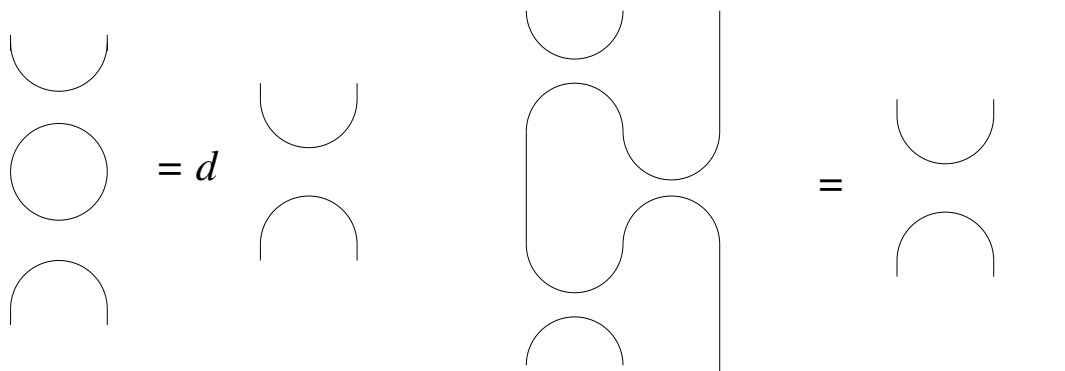Then let the braid $B_i$ be related to $e_i$ by

$$B_i = I - q e_i \qquad\qquad B_i^{-1} = I - q^{-1} e_i$$

for some parameter $q$.

The $B_i$ defined this way satisfy the braid-group relation because the $e_i$ satisfy the

Temperley-Lieb algebra

$$e_i^2 = de_i \qquad\qquad e_i\, e_{i\pm1}\, e_i = e_i$$



where $d = q + q^{-1}$. Note that closed loops are weighted by $d$.

When

$$d = 2\cos[\pi/(k+2)] \qquad \text{i.e.} \quad q = e^{i\pi/(k+2)},$$

these are the statistics of (doubled) $SU(2)_k$ Chern-Simons theory
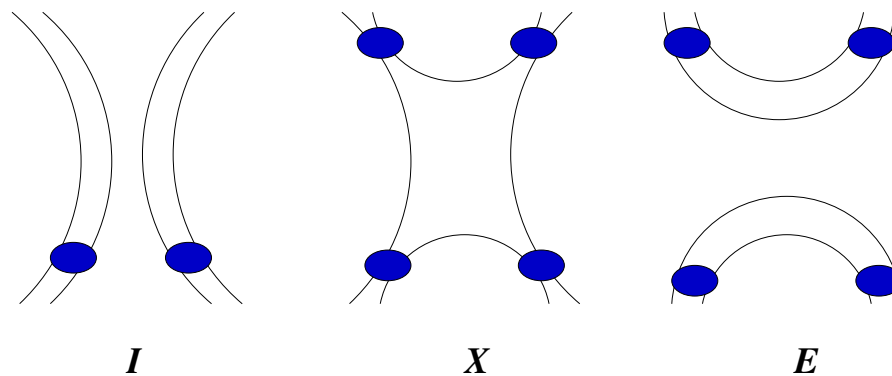
Freedman, Nayak, Shtengel, Walker and Wang

I'm going to discuss the (in some ways simpler) $O(3)_k$ Chern-Simons theory.

For $SU(2)_k$, the particles (Wilson loops) are in the "spin-1/2" representation. For $SO(3)_k$, they are in the "spin-1" representation. We can "fuse" together the spin-1/2 particles to get spin-1 particles by $I - e/d$:



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The braid is then

$$B = q^2 I - X + q^{-2} E$$



In this case, one can check that closed loops get a weight

$$d^2 - 1 = 1 + q^2 + q^{-2}$$

The problem: find a quantum Hamiltonian acting on a two-dimensional Hilbert space which has the above properties.

One answer: Strongly-coupled Yang-Mills theory with a Chern-Simons term

Witten; Ardonne, Fendley and Fradkin

$$S = S_{CS} + S_{SC}$$

$$S_{CS} = \frac{k}{4\pi} \int_M \epsilon^{\mu\nu\alpha} \text{Tr} \left[ A_\mu \partial_\nu A_\alpha + \frac{2}{3} A_\mu A_\nu A_\alpha \right]$$

$$S_{SC} = \frac{1}{2e^2} \int_M \text{Tr} \left[ F_{0i} F^{0i} \right]$$

The ground states are Wilson loops.

The excited states are Polyakov loops.

This is not completely satisfactory: we don't know how to compute much outside of the topological limit, there is no obvious lattice model, and we don't know if a quantum critical point separates this phase from an ordered phase.

The trick: Think of the basis elements of the Hilbert space as states in a classical 2d theory. Then find a quantum Hamiltonian which ground-state wavefunction

$$\Psi_0(s) = \frac{e^{-\beta E_s}}{Z}$$

where the state $s$ has energy $E_s$, and $Z$ is the classical 2d partition function

$$Z = \int_s e^{-\beta E_s - \beta E_s^*}$$

Equal-time correlators in the quantum ground state are classical 2d correlators

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{Z} \int_s \phi_1 \phi_2 \, e^{-\beta E_s - \beta E_s^*}$$

Note that we need to weight configurations by $\left| \Psi_0 \right|^2$. Rokhsar and Kivelson

Our planar projection suggests we look for a quantum loop gas, where the basis states of the two-dimensional Hilbert space are closed loops. In the $SU(2)_k$ and $SO(3)_k$ cases, we want them to have weights $d$ and $d^2 - 1$ respectively.

To find the 2d classical model, let's think instead in terms of a 1+1d quantum model. The loops are the world lines of the particles of the 1+1d theory.

The upshot: Just think of the 2+1d world lines projected down to 1+1d.

We need to ensure that the world lines have the right braiding.

In 1+1d, particles can't go around each other.
1+1d "braiding" is given by the $S$ matrix !

It's well-known from knot theory that if $S(\theta)$ obeys the Yang-Baxter equation, then
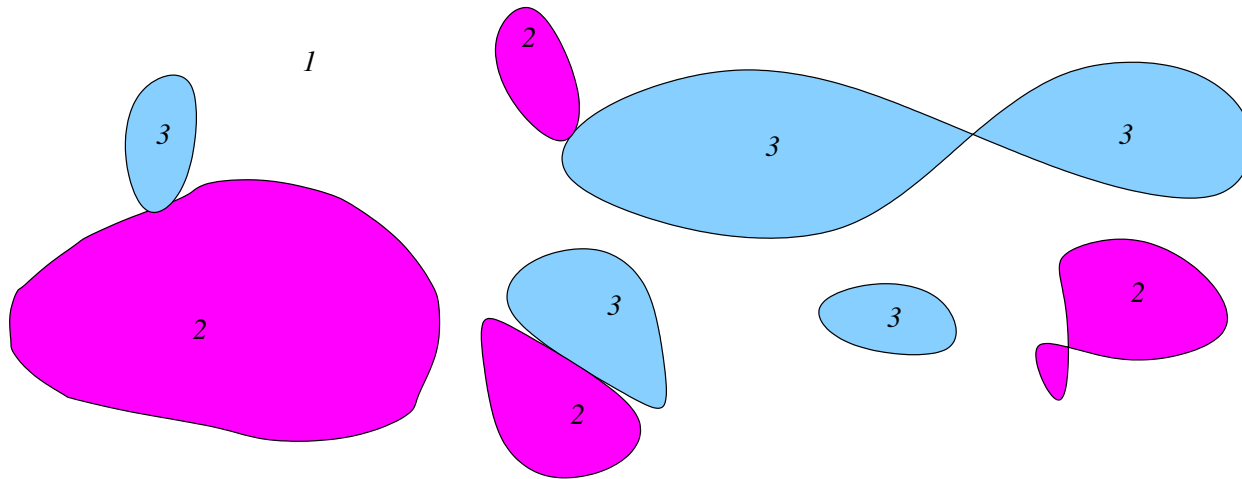
Akutsu, Deguchi and Wadati

$$B = \lim_{\theta \to \infty} S(\theta), \qquad\qquad B^{-1} = \lim_{\theta \to \infty} S(-\theta)$$

where $\theta$ is the rapidity difference of the two particles.

The braiding of the $SO(3)_k$ Chern-Simons theory corresponds to the scattering of the $Q$-state Potts model with

$$Q = d^2 = (q + q^{-1})^2 = 4\cos^2\left(\frac{\pi}{k+2}\right)$$

The weight of $d^2 - 1 = Q - 1$ per loop is the number of different domain walls between the Potts spins.

The Hilbert space of the $SO(3)_k$ quantum loop gas is given by the configurations of the $Q$-state Potts field theory.

This yields a topological field theory when the weight per loop is independent of its length. This occurs at infinite temperature in the 2d classical model.

In the Ising case $Q = 2$ (weight $1$ per loop), this reduces to Kitaev's $\mathbb{Z}_2$ model.

For non-integer $Q$, the $S$ matrices describe scattering of "restricted" kinks in a potential with multiple minima.

Smirnov; Chim and Zamolodchikov; Fendley and Read

These braid matrices obey the Jones-Wenzl projector automatically.

2d classical lattice models with this $S$ matrix and domain walls with weight $4\cos^2[\pi/(k+2)] - 1$ behavior are called dilute $A_k$ models.

Warnaar, Nienhuis, and Seaton

For example, in the dilute $A_3$ model, there are three spins $1, 2, 3$, with the restriction that state $1$ cannot be next to $3$ (a RSOS model).

The Boltzmann weights are such that only regions of $1$ and $2$ are minima. Thus there are two kinds of domain walls: between $1$ and $2$, and between $2$ and itself (spin $3$)
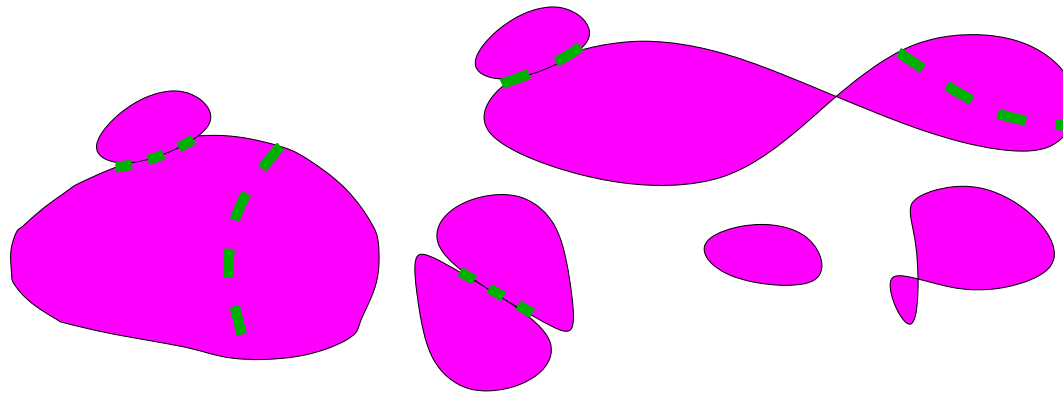


Because of the restriction, the "number" of different domain walls is
$(1 + \sqrt{5})/2 = 2\cos(\pi/5) = 4\cos^2(\pi/5) - 1.$

To determine the phase diagram, remember that a configuration $s$ is weighted by $|\psi(s)|^2$ in the quantum theory.

Thus each weight is squared: each loop gets a weight $(Q-1)^2$.

This suggests that the phase diagram is that of the $Q_{eff}$-state Potts model, where

$$Q_{eff} - 1 = (Q-1)^2 = (d^2 - 1)^2 = 1 + 2\cos[2\pi/(k+2)]$$

There is a critical point when $Q_{eff} \leq 4$: $k = 1, 2, 3$. $k = 1$ is trivial, $k = 2$ is abelian. $k = 3$ is the "Lee-Yang" theory (the braiding rules are those of the Lee-Yang CFT)

The critical point with

$$Q_{eff} = 1 + \left( \left( \frac{1 + \sqrt{5}}{2} \right)^2 - 1 \right)^2 = \frac{5 + \sqrt{5}}{2}$$

is the conformal field theory with $c = 14/15$.

$G_2$ coset !?!

This determines the equal-time correlators in the ground state of the quantum loop gas.

- There are lattice models and field theories which exhibit topological order and conformal quantum critical points. For $SO(3)_k$, Potts; for $SU(2)_k$, $O(n)$ model.

- Equal-time correlators at the critical points can be computed exactly.

- There is a gapped field theory with Chern-Simons topological field theory describing the ground state.

- The excitations of this theory obey non-abelian statistics.

these transparencies at `http://rockpile.phys.virginia.edu/montauk.pdf`